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Gravitational Wave Induced Vibrations of Slender Structures in Space

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Abstract

This paper explores the interaction of weak gravitational fields with slender elastic materials in space and estimates their sensitivities for the detection of gravitational waves with frequencies between 10^{-4} and 1 Hz. The dynamic behaviour of such slender structures is ideally suited to analysis by the simple theory of Cosserat rods. Such a description offers a clean conceptual separation of the vibrations induced by bending, shear, twist and extension and the response to gravitational tidal accelerations can be reliably estimated in terms of the constitutive properties of the structure. The sensitivity estimates are based on a truncation of the theory in the presence of thermally induced homogeneous Gaussian stochastic forces.

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I. INTRODUCTION

Gravitational waves are thought to be produced by astrophysical phenomena ranging from the coalescence of orbiting binaries to violent events in the early Universe. Their detection would herald a new window for the observation of natural phenomena. Great ingenuity is being exercised in attempts to detect such waves in the vicinity of the earth using either laser interferometry or various resonant mass devices following Weber's pioneering efforts with aluminum cylinders. Due to the masking effects of competing influences and the weakness of gravitation compared with the electromagnetic interactions the threshold for the detection of expected gravitationally induced signals remains tantalisingly close to the limits set by currently technology. In order to achieve the signal to noise ratios needed for the unambiguous detection of gravitational waves numerous alternative strategies are also under consideration. These include more sophisticated transducer interfaces, advanced filtering techniques and the use of dedicated arrays of antennae. Earth based gravitational wave detectors require expensive vibration insulation in order to discriminate the required signals from the background. This is one reason why the use of antennae in space offer certain advantages. It is argued here that the gravitationally induced elastodynamic vibrations of slender material structures in space offer other advantages that do not appear to have been considered. Multiple structures of such continua possess attractive properties when used as coincidence detectors of gravitational disturbances with a dominant spectral content in the 10^{-4} to 1 Hz region. Furthermore this window can be readily extended to lower frequencies and higher sensitivities by enlarging the size of the structures.

Newtonian elastodynamics [1] is adequate as a first approximation if supplemented by the *tidal stresses* generated by the presence of spacetime curvature that is small in comparison with the size of the detector. The latter are estimated from the accelerations responsible for spacetime geodesic deviations. Since the constituents of material media owe their elasticity to primarily non-gravitational forces their histories are non-geodesic. The geodesic motions of particles offer a reference configuration and the geodesic deviation of neighbours in a geodesic reference frame provide accelerations that are additionally resisted in a material held together by elastic forces. Since in practical situations the re-radiation of gravitational waves is totally negligible the computation of the stresses induced by the tidal tensor of a background incident gravitational wave offers a viable means of exploring the dynamical response of a material domain to a fluctuating gravitational field.

Current resonant mode detectors are designed to permit reconstruction of the direction and polarisation of gravitational waves that can excite resonances [2]. Clearly such detectors are designed to respond to a narrow spectral window of gravitational radiation and are not particularly good at determining the temporal profile of incident gravitational pulses. A significant advantage of space-based antennae based on slender material structures is that they can be designed to respond to transient gravitational pulses, to polarised uni-directional gravitational waves or omni-directional unpolarised waves.

The general mathematical theory of non-linear Newtonian elasticity is well established. The general theory of one-dimensional Newtonian Cosserat continua derived as limits of three-dimensional continua can be consulted in [1]. The theory is fundamentally formulated in the Lagrangian picture in which material elements are labelled by s. The behaviour of a Cosserat rod at time t may be described in terms of the motion $\boldsymbol{R}(s,t)$ in space of the line of centroids of its cross-sections and elastic deformations about that line. Such a rod is modelled mathematically by an elastic space-curve with structure. This structure defines the relative orientation of neighbouring cross-sections along the rod. Specifying a unit vector $\boldsymbol{d}_3$ (which may be identified with the normal to the cross-section) at each point along the rod enables the state of shear to be related to the angle between this vector and the tangent to the space-curve. Specifying a second vector $\boldsymbol{d}_1$ orthogonal to the first vector (thereby placing it in the plane of the cross-section) can be used to encode the state of flexure and twist along its length. Thus a field of two mutually orthogonal unit vectors along the structure provides three continuous dynamical degrees of freedom that, together with the continuous three degrees of freedom describing a space-curve relative to some arbitrary origin in space, define a simple Cosserat model (see Figure 1). It is significant for this approach that the theory includes thermal variables that can be coupled to the dynamical equations of motion, compatible with the laws of thermodynamics. The theory is completed with equations that relate the deformation strains of the structure to the elastodynamic forces and torques. The simplest constitutive model to consider is based on Kirchoff relations with shear deformation and viscoelasticity. Such a Cosserat model provides a well defined six dimensional quasi-linear hyperbolic system of (integro-)partial differential equations in two independent variables. It may be applied to the study of gravitational wave interactions by suitably choosing external body forces $\boldsymbol{f}$ to represent the tidal interaction with each element in the medium.

A typical system might consist of at least two loops orbiting in interplanetary space. Each structure would be composed of high Q material, several km in length. Such a structure, made up of transportable segments, could be conveyed to an orbiting station and constructed in space. The lowest quadrupole excitation of a steel circular structure would be about 1 Hz and vary inversely as its (stress-free) length. Actuator and feedback instrumentation could be placed around the antennae to "tune" the system to an optimal reference configuration. The precise details of the density and elastic moduli needed to enhance the sensitivity of the receiver would result from an in-depth analytic analysis of the Cosserat equations for free motion. The ability to readily optimise the resonant behaviour for coupled axial, lateral and torsional vibrations by design is a major advantage over other mechanical antennae that have been proposed.

By contrast a broad band detector could consist of an open ended structure coiled into a spiral. For planar spirals with traction free open ends the spectral density of normal transverse and axial linearised modes increases with the density of the spiral winding number. They form ideal broad band detectors with directional characteristics. Furthermore by coupling such a spiral at its outer end to a light mass by a short length of high-Q fibre (such as sapphire) one may tune such an extension to internal resonances, thereby amplifying the spiral elastic excitation. Such excitations offer new detection mechanisms based on the enhanced motion of the outer structure of the spiral.

Cosserat Equations

The dynamical evolution of a rod with mass density, $s \in [0, L] \mapsto \rho(s)$, and cross-sectional area, $s \in [0, L] \mapsto A(s)$, is governed by Newton's dynamical laws:

$$\rho A\,\ddot{\boldsymbol{r}} = \boldsymbol{n}' + \boldsymbol{f}$$

$$\partial_t(\rho\,\boldsymbol{I}(\boldsymbol{w})) = \boldsymbol{m}' + \boldsymbol{r}' \times \boldsymbol{n} + \boldsymbol{l}$$

applied to a triad of orthonormal vectors: $s \in [0, L] \mapsto \{\boldsymbol{d}_1(s,t), \boldsymbol{d}_2(s,t), \boldsymbol{d}_3(s,t)\}$ over the space-curve: $s \in [0, L] \mapsto \boldsymbol{r}(s,t)$ at time t where $\boldsymbol{n}' = \partial_s \boldsymbol{n}$, $\dot{\boldsymbol{r}} = \partial_t \boldsymbol{r}$, $\boldsymbol{f}$ and $\boldsymbol{l}$ denote external force and torque densities respectively and $s \in [0, L] \mapsto \rho\boldsymbol{I}$ is a rod moment of inertia tensor. In these field equations the *contact forces* $\boldsymbol{n}$ and *contact torques* $\boldsymbol{m}$ are related to the vector $\boldsymbol{w}$ and the *strains* $\boldsymbol{u}$, $\boldsymbol{v}$ by constitutive relations. These vectors are themselves defined in terms of the configuration variables $\boldsymbol{r}$ and $\boldsymbol{d}_k$ for $k = 1, 2, 3$ by the relations:

$$\boldsymbol{r}' = \boldsymbol{v}$$

$$\boldsymbol{d}_k' = \boldsymbol{u} \times \boldsymbol{d}_k$$

$$\dot{\boldsymbol{d}}_k = \boldsymbol{w} \times \boldsymbol{d}_k.$$

The latter ensures that the triad remains orthonormal under evolution. The last equation identifies

$$\boldsymbol{w} = \frac{1}{2} \sum_{k=1}^{3} \boldsymbol{d}_k \times \dot{\boldsymbol{d}}_k$$

with the local angular velocity vector of the director triad. The general model accommodates continua whose characteristics (density, cross-sectional area, rotary inertia) vary with s. For a system of two coupled continua with different elastic characteristics on $0 \leq s < s_0$ and $s_0 < s \leq L$ respectively one matches the degrees of freedom at $s = s_0$ according to a junction condition describing the coupling.

To close the above equations of motion constitutive relations appropriate to the rod must be specified: $\boldsymbol{n}(s,t) = \hat{\boldsymbol{n}}(\boldsymbol{u}(s,t), \boldsymbol{v}(s,t), \boldsymbol{u}_t(s), \boldsymbol{v}_t(s), \ldots, s)$, $\boldsymbol{m}(s,t) = \hat{\boldsymbol{m}}(\boldsymbol{u}(s,t), \boldsymbol{v}(s,t), \boldsymbol{u}_t(s), \boldsymbol{v}_t(s), \ldots, s)$ where $\boldsymbol{u}_t(s)$ etc., denote the history of $\boldsymbol{u}(s,t)$ up to time t. These relations specify a reference configuration (say at $t = 0$) with strains $\boldsymbol{U}(s), \boldsymbol{V}(s)$ such that $\hat{\boldsymbol{n}}(\boldsymbol{U}(s), \boldsymbol{V}(s), \ldots, s)$ and $\hat{\boldsymbol{m}}(\boldsymbol{U}(s), \boldsymbol{V}(s), \ldots, s)$ are specified. A reference configuration free of flexure has $\boldsymbol{r}_s(s,0) = \boldsymbol{d}_3(s,0)$, i.e. $\boldsymbol{V}(s) = \boldsymbol{d}_3(s,0)$. If a standard configuration is such that $\boldsymbol{r}(s,0)$ is a space-curve with Frenet curvature κ_0 and torsion τ_0 and the standard directors are oriented so that $\boldsymbol{d}_1(s,0)$ is the unit normal to the space-curve and $\boldsymbol{d}_2(s,0)$ the associated unit binormal then $\boldsymbol{U}(s) = \kappa_0(s)\, \boldsymbol{d}_2(s,0) + \tau_0(s)\, \boldsymbol{d}_3(s,0)$.

For a rod of density ρ and cross-sectional area A in a weak plane gravitational wave background the simplest model to consider consists of the Newtonian Cosserat equations above with a time dependent body force modelled by a gravitational tidal interaction $\boldsymbol{f}$. In addition to time-dependent waves this may include stationary Newtonian gravitational fields. These add terms of the form $\rho A \tilde{\boldsymbol{g}}$ to $\boldsymbol{f}$ where $\tilde{\boldsymbol{g}}$ is the "effective local acceleration due to gravity". Post Newtonian gravitational fields (such as gravitomagnetic and Lens-Thirring effects) can be accommodated with a more refined metric background.

An important consideration of any modelling of Cosserat continua to low levels of excitation is the estimation of signal to noise ratios induced by anelasticity and thermal interactions. To gain an insight into the former one may attempt to extend the established theory of linear anelasticity to a Cosserat structure. For a string with uniform density ρ, static Young's modulus E and area of cross section A, the free damped motion in one dimension is modelled by the equation:

$$\rho A\, \partial_{tt} x(s,t) = n'(s,t)$$

where the axial strain $v(x,t) = \partial_s x(s,t) \equiv x'(s,t)$ and

$$n(s,t) = EA(v(s,t)-1) - EA \int_{-\infty}^{t} \phi(t-t')\, \dot{v}(s,t')dt'$$

for some viscoelastic model ϕ with $0 \leq s \leq L$. For free motion in the mode:

$$x(s,t) = s + \xi(t)\cos(\pi s/L)$$

the amplitude $\xi(t)$ satisfies

$$\ddot{\xi}(t) + \omega_0^2\, \xi(t) = \omega_0^2 \int_{-\infty}^{t} \phi(t-t')\dot{\xi}(t')dt'$$

with $\omega_0^2 = \frac{\pi^2 E}{L^2 \rho}$ while for a forced harmonic excitation:

$$\ddot{\xi}(t) + \omega_0^2 \xi(t) = \omega_0^2 \int_{-\infty}^{t} \phi(t-t')\dot{\xi}(t')dt' + F_0 \exp(-i\Omega t).$$

With $\xi(0) = \dot{\xi}(0) = 0$ the Laplace transformed amplitude of forced axial motion is then given in terms of the Laplace transform [10]

$$\bar{\phi}(\sigma) = \int_0^{\infty} \phi(t) e^{-\sigma t} dt$$

of the anelastic modelling function $\phi(t)$ as:

$$\bar{\xi}(\sigma) = \frac{F_0}{(\sigma - i\Omega)(\sigma^2 - \omega_0^2 \sigma \bar{\phi}(\sigma) + \omega_0^2)}.$$

To extend this approach in a simple manner to a 3-D Cosserat model of a slender rod with uniform static moduli E and G, geometric elements A, $K_{\alpha\alpha} = J_{11} + J_{22}$, $J_{\alpha\beta}$, one adopts the following constitutive relations for the local director components of the contact force $\boldsymbol{n}$ and torque $\boldsymbol{m}$ in terms of the local strain vectors $\boldsymbol{v}$ and $\boldsymbol{u}$ and anelastic response functions ϕ_E and ϕ_G:

$$n_3(s,t) = EA\,(v_3(s,t) - 1) - EA \int_{-\infty}^{t} \phi_E(t-t')\,\dot{v}_3(s,t')\,dt'$$

$$n_1(s,t) = GA\,v_1(s,t) - GA \int_{-\infty}^{t} \phi_G(t-t')\,\dot{v}_1(s,t')\,dt'$$

$$n_2(s,t) = GA\,v_2(s,t) - GA \int_{-\infty}^{t} \phi_G(t-t')\,\dot{v}_2(s,t')\,dt'$$

$$m_3(s,t) = K_{\alpha\alpha}G\,u_3(s,t) - K_{\alpha\alpha}G \int_{-\infty}^{t} \phi_G(t-t')\,\dot{u}_3(s,t')\,dt'$$

$$m_\alpha(s,t) = E\sum_{\beta=1}^{2} J_{\alpha\beta}\,u_\beta(s,t) - E\sum_{\beta=1}^{2} J_{\alpha\beta} \int_{-\infty}^{t} \phi_E(t-t')\,\dot{u}_\beta(s,t')\,dt'$$

for $\alpha, \beta = 1, 2$.

II. BEHAVIOUR OF AN ELASTIC STRING IN A NOISY WEAK GRAVITATIONAL WAVE BACKGROUND

The above outlines a new approach to the modelling of gravitational interactions with slender structures in space. Given the topology and material properties of an antenna one may analyse its response to such signals in terms of solutions to a deterministic system of well defined partial (integro-)differential equations. These equations are in general easier to analyse than those describing the elastodynamics of three-dimensional materials. However in addition to controlling the temperature dependence of material characteristics, thermal interactions with such structures will induce a stochastic element into the signal response. It is therefore necessary to seek modifications to the above Cosserat description that can accommodate such random interactions. In the presence of internal damping this is non-trivial problem for a broad band resonant detector.

In the calculation that follows we shall oversimplify this problem in order to gain some order of magnitude estimates of signal-to-(thermal)noise ratios for both narrow and broad band resonant detectors made of known high-Q materials. The simplest approach is to approximate the Cosserat equations by ignoring flexure and torsional mode excitations and explore the purely string-like excitations. Commensurate with these approximations we shall assume that the damping can be described in terms of a single Q-factor at about 4 K for each resonant mode and that the thermal interactions give rise to a stochastic system driven by a spatially homogeneous Gaussian noise term.

We shall make two further restrictions by assuming that the antenna possesses sufficient stiffness to maintain planar motion in tension (e.g. by overall slow rotation about its centre of mass) throughout the excitation time and that the environment produces no overall translational drift away from a regular orbital motion of its centroid. The plane will be chosen orthogonal to the direction of propagation of a gravitational disturbance.

We therefore begin by examining the linearised modes about an arbitrary planar 1-dimensional structure. In this way we can address the response of both a narrow band planar loop and a broad band planar spiral together. Given the anelastic characteristics of the structure all of these restrictions can be readily relaxed at the cost of increased complexity in the thermo-mechanical analysis.

In such a framework consider the (inertial) Cartesian basis $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ with corresponding coordinates (x, y, z) such that the tidal acceleration field at any position $\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$ and time t, corresponding to a plane gravitational wave travelling in the direction $\boldsymbol{k}$ at the speed of light c has the form [3]

$$\boldsymbol{g}(\boldsymbol{r}, t) \;=\; \left\{ \ddot{\mathcal{A}}_1(t - z/c)\, x + \ddot{\mathcal{A}}_2(t - z/c)\, y \right\} \frac{\boldsymbol{i}}{2} + \left\{ \ddot{\mathcal{A}}_2(t - z/c)\, x - \ddot{\mathcal{A}}_1(t - z/c)\, y \right\} \frac{\boldsymbol{j}}{2}. \quad (1)$$

The dimensionless waveforms $\mathcal{A}_1(t - z/c)$ and $\mathcal{A}_2(t - z/c)$ arise from two independent polarisations of the gravitational wave in the transverse-traceless gauge in linearised gravitation. This acceleration excites an elastic string with reference length L, cross-section area A and mass density ρ described by a space-curve $\boldsymbol{r}(s, t)$ parameterised by the parameter $s \in [0, L]$ according to

$$\rho A\, \ddot{\boldsymbol{r}}(s, t) - \boldsymbol{n}'(s, t) \;=\; \rho A\, \boldsymbol{g}(\boldsymbol{r}(s, t), t). \quad (2)$$

The contact force (tension) $\boldsymbol{n}(s, t)$ is given by the constitutive relation:

$$\boldsymbol{n}(s, t) \;=\; EA\left(|\boldsymbol{r}'(s, t)| - 1\right) \frac{\boldsymbol{r}'(s, t)}{|\boldsymbol{r}'(s, t)|} \quad (3)$$

in terms of the Young's modulus E. For a closed string, the periodic boundary conditions are

$$\boldsymbol{r}(0,t) \;=\; \boldsymbol{r}(L,t) \tag{4}$$

$$\boldsymbol{r}'(0,t) \;=\; \boldsymbol{r}'(L,t). \tag{5}$$

For an open string we consider the tension-free boundary conditions given by

$$\boldsymbol{n}(0,t) \;=\; \boldsymbol{n}(L,t) = 0. \tag{6}$$

Let $\boldsymbol{r}_0(s)$ be a stress-free *reference configuration* of the elastic string with a unit stretch, i.e. $|\boldsymbol{r}_0'(s)| = 1$. Assuming the Frenet curvature of $\boldsymbol{r}_0(s)$, given by

$$\kappa_0(s) \;\stackrel{\text{def}}{=}\; |\boldsymbol{r}_0''(s)| \tag{7}$$

is non-vanishing introduce a Frenet frame as the set of orthonormal vectors along the string: [5]

$$\boldsymbol{t}_0(s) \;\stackrel{\text{def}}{=}\; \boldsymbol{r}_0'(s) \tag{8}$$

$$\boldsymbol{a}_0(s) \;\stackrel{\text{def}}{=}\; \frac{\boldsymbol{t}_0'(s)}{\kappa_0(s)} \tag{9}$$

$$\boldsymbol{b}_0(s) \;\stackrel{\text{def}}{=}\; \boldsymbol{t}_0(s) \times \boldsymbol{a}_0(s). \tag{10}$$

The space-curve of the reference configuration $\boldsymbol{r}_0(s)$ is defined in the x-y plane with $z = 0$ so that

$$\boldsymbol{r}_0(s) \;=\; x_0(s)\,\boldsymbol{i} + y_0(s)\,\boldsymbol{j}. \tag{11}$$

With

$$\boldsymbol{b}_0 \;=\; \boldsymbol{k} \tag{12}$$

it follows [5] that the tangent vector $\boldsymbol{t}_0(s)$ may be expressed in terms of the Frenet curvature of the reference space-curve as

$$\boldsymbol{t}_0(s) \;=\; -\sin\left(\int_0^s \kappa_0(s')\,\mathrm{d}s' + \theta\right)\boldsymbol{i} + \cos\left(\int_0^s \kappa_0(s')\,\mathrm{d}s' + \theta\right)\boldsymbol{j} \tag{13}$$

for an arbitrary constant θ. Let $\alpha(s,t)$ and $\beta(s,t)$ be axial and transverse perturbations in the x-y plane of the string about $\boldsymbol{r}_0(s)$ so that

$$\boldsymbol{r}(s,t) \;=\; \boldsymbol{r}_0(s) + \alpha(s,t)\,\boldsymbol{t}_0(s) + \beta(s,t)\,\boldsymbol{a}_0(s). \tag{14}$$

Since the interaction between the string and the gravitational wave is assumed small we shall drop terms containing powers of $\alpha, \beta, \mathcal{A}_1, \mathcal{A}_2$ higher than 1 in the following. Thus (1) becomes

$$\begin{aligned}\boldsymbol{g}(\boldsymbol{r}(s,t),t) \;=\;& \left\{\ddot{\mathcal{A}}_1(t)\,F_1(s) + \ddot{\mathcal{A}}_2(t)\,F_2(s)\right\}\boldsymbol{t}_0(s)\\ &+\left\{\ddot{\mathcal{A}}_2(t)\,F_1(s) - \ddot{\mathcal{A}}_1(t)\,F_2(s)\right\}\boldsymbol{a}_0(s)\end{aligned} \tag{15}$$

up to first order terms in α and β, where

$$F_1(s) \;\stackrel{\text{def}}{=}\; \frac{1}{4}\left\{x_0(s)^2 - y_0(s)^2\right\}' \tag{16}$$

$$F_2(s) \;\stackrel{\text{def}}{=}\; \frac{1}{2}\left\{x_0(s)\,y_0(s)\right\}'. \tag{17}$$

¿From (8), (9), (10) and (12)

$$\boldsymbol{r}' \;=\; (1 + \alpha' - \kappa_0\,\beta)\,\boldsymbol{t}_0 + (\alpha\,\kappa_0 + \beta')\,\boldsymbol{a}_0 \tag{18}$$

and therefore

$$\boldsymbol{n}' \;=\; EA\,(\alpha' - \kappa_0\,\beta)'\,\boldsymbol{t}_0 + EA\,\kappa_0\,(\alpha' - \kappa_0\,\beta)\,\boldsymbol{a}_0 \tag{19}$$

up to first order terms in α and β. Substituting this into (2) to first order yields

$$\ddot{\alpha}(s,t) - \frac{E}{\rho}\,\left\{\alpha'(s,t) - \kappa_0(s)\,\beta(s,t)\right\}' \;=\; \ddot{\mathcal{F}}_\alpha(s,t) \tag{20}$$

$$\ddot{\beta}(s,t) - \frac{E}{\rho}\,\kappa_0(s)\,\left\{\alpha'(s,t) - \kappa_0(s)\,\beta(s,t)\right\} \;=\; \ddot{\mathcal{F}}_\beta(s,t) \tag{21}$$

where

$$\mathcal{F}_\alpha(s,t) \;\stackrel{\text{def}}{=}\; \mathcal{A}_1(t)\,F_1(s) + \mathcal{A}_2(t)\,F_2(s) \tag{22}$$

$$\mathcal{F}_\beta(s,t) \;\stackrel{\text{def}}{=}\; \mathcal{A}_2(t)\,F_1(s) - \mathcal{A}_1(t)\,F_2(s). \tag{23}$$

For an open string the tension-free condition (6) becomes

$$\alpha'(0,t) - \kappa_0(0)\,\beta(0,t) = \alpha'(L,t) - \kappa_0(L)\,\beta(L,t) \;=\; 0. \tag{24}$$

Introducing

$$\chi(s,t) \;\stackrel{\text{def}}{=}\; \alpha(s,t) - \left(\frac{\beta(s,t)}{\kappa_0(s)}\right)' \tag{25}$$

and using (20) and (21) gives

$$\ddot{\chi}(s,t) \;=\; \ddot{\mathcal{F}}_\alpha(s,t) - \left(\frac{\ddot{\mathcal{F}}_\beta(s,t)}{\kappa_0(s)}\right)' \tag{26}$$

which implies

$$\chi(s,t) \;=\; \mathcal{F}_\alpha(s,t) - \left(\frac{\mathcal{F}_\beta(s,t)}{\kappa_0(s)}\right)' + \chi_0(s) + \chi_1(s)\,t \tag{27}$$

for two arbitrary functions $\chi_0(s)$ and $\chi_1(s)$. Furthermore by introducing

$$\xi(s,t) \;\stackrel{\text{def}}{=}\; \frac{\beta(s,t) - \mathcal{F}_\beta(s,t)}{L\,\kappa_0(s)} \tag{28}$$

and substituting (25) and (27) into (21) we obtain

$$\ddot{\xi}(s,t) - \frac{E}{\rho}\,\left\{\xi''(s,t) - \kappa_0(s)^2\,\xi(s,t)\right\} \;=\; \frac{1}{L}\,\frac{E}{\rho}\,\{\mathcal{F}(s,t) + \chi_0'(s) + \chi_1'(s)\,t\} \tag{29}$$

where

$$\mathcal{F}(s,t) \stackrel{\text{def}}{=} \mathcal{F}_\alpha'(s,t) - \kappa_0(s)\,\mathcal{F}_\beta(s,t) \;=\; \mathcal{A}_1(t)\,\mathcal{F}_1(s) + \mathcal{A}_2(t)\,\mathcal{F}_2(s) \tag{30}$$

in terms of

$$\mathcal{F}_1(s) \;\stackrel{\text{def}}{=}\; F_1'(s) + \kappa_0(s)\,F_2(s) = -\frac{1}{2}\,\cos\left(2\mathcal{K}_0(s) + \theta\right) \tag{31}$$

$$\mathcal{F}_2(s) \;\stackrel{\text{def}}{=}\; F_2'(s) - \kappa_0(s)\,F_1(s) = -\frac{1}{2}\,\sin\left(2\mathcal{K}_0(s) + \theta\right) \tag{32}$$

with

$$\mathcal{K}_0(s) \;\stackrel{\text{def}}{=}\; \int_0^s \kappa_0(s')\,\mathrm{d}s'. \tag{33}$$

In deriving the above relations, (13), (16) and (17) have been used. It follows from (25), (27) and (28) that

$$\alpha(s,t) \;=\; L\,\xi'(s,t) + \mathcal{F}_\alpha(s,t) + \chi_0(s) + \chi_1(s)\,t \tag{34}$$

$$\beta(s,t) \;=\; L\,\kappa_0(s)\xi(s,t) + \mathcal{F}_\beta(s,t). \tag{35}$$

For an open string with tension-free ends we may substitute (34) and (35) into (24), (25) to obtain

$$\ddot{\xi}(0,t) = \ddot{\xi}(L,t) \;=\; 0 \tag{36}$$

by using (29). Therefore $\alpha(s,t)$ and $\beta(s,t)$ can be obtained by solving (29) for a choice of functions $\chi_0(s)$ and $\chi_1(s)$, subject to the boundary condition (36) for an open string or

$$\xi(0,t) \;=\; \xi(L,t) \tag{37}$$

$$\xi'(0,t) \;=\; \xi'(L,t) \tag{38}$$

together with $\chi_0(0) = \chi_0(L), \chi_0'(0) = \chi_0'(L)$ and $\chi_1(0) = \chi_1(L), \chi_1'(0) = \chi_1'(L)$ for a closed string.

III. NORMAL MODE ANALYSIS

We next analyse (20) and (21) for displacement perturbations that remain small compared with L at all times. These solutions represent the deterministic dynamic response of a string in terms of small oscillatory deviations from a time-independent reference configuration $\boldsymbol{r}_0(s)$ under the influence of weak gravitational waves. Such $\alpha(s,t)$ and $\beta(s,t)$ can be obtained by choosing

$$\chi_0(s) = \chi_1(s) \;=\; 0 \tag{39}$$

that enter (29), (34) and (35). For an open string we further choose

$$\xi(0,t) = \xi(L,t) \;=\; 0 \tag{40}$$

as solutions to (36). To proceed we express $\xi(s,t)$ in terms of normal modes, a complete set of basis functions $\{\psi_p(s)\}$ indexed by (integer- or multi-integer valued) p satisfying

$$\psi_p''(s) + \left\{\frac{\rho}{E}\,\omega_p^2 - \kappa_0(s)^2\right\}\psi_p(s) \;=\; 0 \tag{41}$$

subject to either periodic or tension-free ($\psi_p(0) = 0$, $\psi_p(L) = 0$) boundary conditions with the corresponding eigen-values $\{\frac{\rho}{E}\,\omega_p^2\}$. In addition the eigen-functions $\psi_p(s)$ shall satisfy the ortho-normality conditions

$$\int_0^L \psi_p(s)\,\psi_q(s)\,\mathrm{d}s \;=\; L\,\delta_{pq}. \tag{42}$$

In this basis

$$\xi(s,t) \;=\; \sum_p \xi_p(t)\,\psi_p(s) \tag{43}$$

summing over the full range of p.

Substituting (39) and (43) into (29) and using (41) and (42) gives

$$\ddot{\xi}_p(t) + \omega_p^2\,\xi_p(t) \;=\; f_p(t) \tag{44}$$

where

$$f_p(t) \;\stackrel{\mathrm{def}}{=}\; \frac{1}{L}\,\frac{E}{\rho}\,\{a_{1p}\,\mathcal{A}_1(t) + a_{2p}\,\mathcal{A}_2(t)\} \tag{45}$$

with the "overlap coefficients" given by

$$a_{1p} \;\stackrel{\mathrm{def}}{=}\; \frac{1}{L}\int_0^L \mathcal{F}_1(s)\,\psi_p(s)\,\mathrm{d}s \tag{46}$$

$$a_{2p} \;\stackrel{\mathrm{def}}{=}\; \frac{1}{L}\int_0^L \mathcal{F}_2(s)\,\psi_p(s)\,\mathrm{d}s. \tag{47}$$

It follows from (31) and (32) that these coefficients do not depend on the choice of the origin on the x-y plane. In addition the coefficients

$$a_p \;\stackrel{\mathrm{def}}{=}\; \sqrt{a_{1p}^2 + a_{2p}^2} \tag{48}$$

are independent of θ and invariant under rotation of the x- and y-axes about the z-axis. The value of a_p provides a measure of the "coupling strength" between the p-th-normal mode and the exciting gravitational wave.

IV. MECHANICAL ENERGY IN TERMS OF NORMAL MODES

The mechanical energy of the string satisfying (2) is given by

$$\hat{\mathcal{E}} \;=\; \frac{1}{2}\int_0^L \left\{\rho A\,\dot{\boldsymbol{r}}^2 + EA\left(|\boldsymbol{r}'|-1\right)^2\right\} \mathrm{d}s. \tag{49}$$

Substituting (14) into (49) gives

$$\hat{\mathcal{E}} \;=\; \frac{1}{2}\int_0^L \left\{\rho A\left(\dot{\alpha}^2+\dot{\beta}^2\right) + EA\left(\alpha'-\kappa_0\,\beta\right)^2\right\} \mathrm{d}s \tag{50}$$

up to first order terms in α and β. By using (34) and (35) the above expression becomes

$$\hat{\mathcal{E}} \;=\; \frac{1}{2}\int_0^L \left\{\rho A\left(L^2\,\dot{\xi}'^2 + L^2\,\kappa_0\,\dot{\xi}^2 + \dot{\mathcal{F}}_\alpha^2 + \dot{\mathcal{F}}_\beta^2 - 2\,L\,\dot{\xi}\,\dot{\mathcal{F}}\right) + EA\left(L\,\xi'' - L\,\kappa_0^2\,\xi + \mathcal{F}\right)^2\right\} \mathrm{d}s.$$

In the absence of the gravitational wave excitation this reduces to

$$\mathcal{E} \;=\; \frac{\rho A\,L^2}{2}\int_0^L \left\{\dot{\xi}'^2 + \kappa_0\,\dot{\xi}^2 + \frac{\rho}{E}\,\ddot{\xi}^2\right\} \mathrm{d}s \tag{51}$$

where (29) has been used. From (43), (41) and (42) together with boundary conditions (36) for an open string or (37), (38) for a closed string we obtain

$$\mathcal{E} \;=\; \sum_p \mathcal{E}_p \tag{52}$$

with modal contributions

$$\mathcal{E}_p \;\stackrel{\text{def}}{=}\; \frac{1}{2}\,m_p\left(\omega_p^2\,\xi_p^2 + \dot{\xi}_p^2\right) \tag{53}$$

where

$$m_p \;\stackrel{\text{def}}{=}\; \frac{\rho^2 A\,L^3\,\omega_p^2}{E} \tag{54}$$

may be identified as an effective modal mass.

V. SENSITIVITY ESTIMATION

To accommodate the thermal fluctuations and dissipation for a vibrating string with small displacements, the modal equations (44) should be modified into a set of *coupled linear stochastic equations* of the form

$$\ddot{\xi}_p(t) + \sum_q 2\,\zeta_{pq}\,\omega_q\,\dot{\xi}_q(t) + \omega_p^2\,\xi_p(t) \;=\; f_p(t) + \dot{w}_p(t) \tag{55}$$

where the $\{\zeta_{pq}\}$ are (in general non-diagonal) coupling constants related to material visco-elasticity and the $\{w_p(t)\}$ are independent Wiener processes characterised by Gaussian probability densities:

$$\mathcal{P}(w_p(t) - w_p(t-\Delta t)) \;=\; \frac{1}{\sqrt{2\pi b_p\,\Delta t}}\,\exp\left\{-\frac{[w_p(t) - w_p(t-\Delta t)]^2}{2\,b_p\,\Delta t}\right\} \tag{56}$$

at any time t for some width parameters b_p. As discussed above we assume here that ζ_{pq} for $p \neq q$, vanish identically (*proportional damping* [6]). Then (55) reduces to a system of *decoupled linear stochastic equations*

$$\ddot{\xi}_p(t) + 2\,\zeta_p\,\omega_p\,\dot{\xi}_p(t) + \omega_p^2\,\xi_p(t) \;=\; f_p(t) + \dot{w}_p(t) \tag{57}$$

in terms of the modal damping ratios $\zeta_p \stackrel{\text{def}}{=} \zeta_{pp}$ for any p. These are related to quality factors Q_p and relaxation times τ_p by

$$\zeta_p \;=\; \frac{1}{2Q_p} = \frac{1}{2\omega_p\tau_p}. \tag{58}$$

To maintain thermal equilibrium at a temperature T between fluctuations and dissipation depending on ζ_p and $w_p(t)$ the parameters b_p are given by [7]

$$b_p \;=\; \frac{2\,k_B T}{m_p\tau_p} \tag{59}$$

involving Boltzmann's constant k_B ($= 1.38 \times 10^{-23}$ J/K.)

Given initial conditions for $\xi_p(t)$ and $\dot{\xi}_p(t)$ at $t = 0$, a representative solution of (57) for $t > 0$ may be conveniently expressed as

$$\xi_p(t) \;=\; \xi_{0p}(t) + \xi_{f_p}(t) + \xi_{\dot{w}_p}(t) \tag{60}$$

where

$$\xi_{0p}(t) \;\stackrel{\text{def}}{=}\; \xi_p(0)\sqrt{1-\zeta_p^2}\,\omega_p\,\Theta_p(t) + (\xi_p(0)\,\zeta_p\,\omega_p + \dot{\xi}_p(0))\,\Phi_p(t) \tag{61}$$

$$\xi_{f_p}(t) \;\stackrel{\text{def}}{=}\; \int_0^\infty \Phi_p(t-t')f_p(t')\,\mathrm{d}t' \tag{62}$$

$$\xi_{\dot{w}_p}(t) \;\stackrel{\text{def}}{=}\; \int_0^\infty \Phi_p(t-t')\dot{w}_p(t')\,\mathrm{d}t' \tag{63}$$

in terms of

$$\Theta_p(t) \stackrel{\text{def}}{=} H(t)\,\frac{e^{-\zeta_p\,\omega_p\,t}}{\sqrt{1-\zeta_p^2}\,\omega_p}\,\cos(\sqrt{1-\zeta_p^2}\,\omega_p\,t) \tag{64}$$

$$\Phi_p(t) \stackrel{\text{def}}{=} H(t)\,\frac{e^{-\zeta_p\,\omega_p\,t}}{\sqrt{1-\zeta_p^2}\,\omega_p}\,\sin(\sqrt{1-\zeta_p^2}\,\omega_p\,t) \tag{65}$$

using the Heaviside function

$$H(t) \stackrel{\text{def}}{=} \begin{cases} 1 \text{ if } t \geq 0 \\ 0 \text{ otherwise.} \end{cases} \tag{66}$$

It follows [7] that the $\xi_{\dot{w}_p}(t)$ given in (63) give rise to the time-dependent deviation values:

$$\langle \xi_{\dot{w}_p}(t)^2 \rangle = \frac{k_B T}{m_p\,\omega_p^2}\left\{1 - e^{-t/\tau_p}\Gamma_+\right\} \tag{67}$$

$$\langle \dot{\xi}_{\dot{w}_p}(t)^2 \rangle = \frac{k_B T}{m_p}\left\{1 - e^{-t/\tau_p}\Gamma_-\right\} \tag{68}$$

where

$$\Gamma_\pm = 1 \pm \frac{\zeta_p}{\sqrt{1-\zeta_p^2}}\sin(2\sqrt{1-\zeta_p^2}\,\omega_p\,t) + \frac{2\zeta_p^2}{1-\zeta_p^2}\sin^2(\sqrt{1-\zeta_p^2}\,\omega_p\,t).$$

For given gravitational waveforms $\mathcal{A}_1(t)$ and $\mathcal{A}_2(t)$ and the corresponding $f_p(t)$ evaluated using (45), the "signal" displacements $\xi_{f_p}(t)$ and their derivatives at any measurement time $t = \tau > 0$ may be calculated using (62) and compared with those due to thermal "noise" given by (67) and (68).

The above noise terms have small variances if $\tau \ll \tau_p$, as pointed out in [8] in analysing a bar-type gravitational wave detector. In this case for "high-Q" materials with $\zeta_p \ll 1$ the ratio of "effective mechanical energy" due to signals to "effective thermal energy" at $t = \tau \ll \tau_p$ may be approximated as follows. Based on (53) introduce the effective mechanical energy associated with the displacement $\xi_{f_p}(t)$ in (62) due to a signal at measurement time τ given by:

$$\mathcal{E}_p^{(S)}(\tau) \stackrel{\text{def}}{=} \frac{1}{2}\,m_p\,\omega_p^2\,\xi_{f_p}(\tau)^2 + \frac{1}{2}\,m_p\,\dot{\xi}_{f_p}(\tau)^2 \tag{69}$$

$$= m_p \int_0^\tau \dot{\xi}_{f_p}(t)\,f(t)\,\mathrm{d}t - 2\,\zeta_p\,\omega_p\,m_p \int_0^\tau \dot{\xi}_{f_p}(t)^2\,\mathrm{d}t. \tag{70}$$

Let

$$f_p^{|\tau}(t) \stackrel{\text{def}}{=} H(\tau - t)\, f_p(t) \tag{71}$$

and denote accordingly

$$\xi_{f_p^{|\tau}}(t) \stackrel{\text{def}}{=} \int_0^\infty \Phi_p(t-t') f_p^{|\tau}(t')\, \mathrm{d}t' \tag{72}$$

so that $\xi_{f_p}(t) = \xi_{f_p^{|\tau}}(t)$ for $0 < t < \tau$. Now

$$\int_0^\tau \dot{\xi}_{f_p}(t)\, f_p(t)\, \mathrm{d}t = \int_0^\infty \dot{\xi}_{f_p^{|\tau}}(t)\, f_p^{|\tau}(t)\, \mathrm{d}t = \frac{1}{2\pi} \Re \int_{-\infty}^\infty i\,\omega\, \widetilde{\xi}_{f_p^{|\tau}}(\omega)\, \widetilde{f}_p^{|\tau}(\omega)^*\, \mathrm{d}\omega \tag{73}$$

in terms of the one-sided Fourier transform operator $\widetilde{}$ and its inverse given by

$$\widetilde{F}(\omega) \stackrel{\text{def}}{=} \int_0^\infty F(t)\, e^{-i\omega t}\, \mathrm{d}t, \quad F(t) = \frac{1}{2\pi} \int_{-\infty}^\infty F(\omega)\, e^{i\omega t}\, \mathrm{d}\omega \tag{74}$$

for any function F of non-negative t. It follows from (72) that

$$\widetilde{\xi}_{f_p^{|\tau}}(\omega) = \widetilde{\Phi}_p(\omega) \widetilde{f}_p^{|\tau}(\omega) \tag{75}$$

where

$$\widetilde{\Phi}_p(\omega) = \frac{1}{\omega_p^2 - \omega^2 + 2\, i\, \zeta_p\, \omega_p\, \omega} \tag{76}$$

and therefore

$$\int_0^\tau \dot{\xi}_{f_p}(t)\, f_p(t)\, \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{2\,\zeta_p\,\omega_p\,\omega^2}{(\omega_p^2 - \omega^2)^2 + 4\,\zeta_p^2\,\omega_p^2\,\omega^2} \left| \widetilde{f}_p^{|\tau}(\omega) \right|^2 \mathrm{d}\omega. \tag{77}$$

For sufficiently small ζ_p

$$\frac{2\,\zeta_p\,\omega_p\,\omega^2}{(\omega_p^2 - \omega^2)^2 + 4\,\zeta_p^2\,\omega_p^2\,\omega^2} \approx \frac{\pi}{2} \{\delta(\omega - \omega_p) + \delta(\omega + \omega_p)\} \tag{78}$$

(in the sense of distributions) and the second term in (70) becomes negligible. This yields

$$\mathcal{E}_p^{(S)}(\tau) \approx \frac{m_p}{2} \left| \widetilde{f}_p^{|\tau}(\omega_p) \right|^2 = \frac{1}{2}\, EAL\, \omega_p^2 \left| a_{1p}\, \widetilde{\mathcal{A}}_1^{|\tau}(\omega_p) + a_{2p}\, \widetilde{\mathcal{A}}_2^{|\tau}(\omega_p) \right|^2. \tag{79}$$

The effective thermal noise energy due to $\xi_{\dot{w}_p}(t)$ in (63) is given by

$$\mathcal{E}_p^{(N)}(\tau) \stackrel{\text{def}}{=} \frac{1}{2}\, m_p\, \omega_p^2\, \langle \xi_{\dot{w}_p}(\tau)^2 \rangle + \frac{1}{2}\, m_p\, \langle \dot{\xi}_{\dot{w}_p}(\tau)^2 \rangle \tag{80}$$

$$\approx 2\,\zeta_p\,\omega_p\,\tau\, k_B T = \frac{\tau}{\tau_p}\, k_B T \tag{81}$$

for $\tau \ll \tau_p$ and $\zeta_p \ll 1$. In these limits a gravitational signal to thermal noise ratio can be estimated to be

$$\frac{\mathcal{E}_p^{(S)}(\tau)}{\mathcal{E}_p^{(N)}(\tau)} \approx \frac{\omega_p^2}{2}\frac{\tau_p}{\tau}\frac{EAL}{k_B T}\left| a_{1p}\,\widetilde{\mathcal{A}}_1^{|\tau}(\omega_p) + a_{2p}\,\widetilde{\mathcal{A}}_2^{|\tau}(\omega_p)\right|^2 . \tag{82}$$

The response of any antennae to a general pulse of gravitational radiation can be estimated once its spectral content is known. Consider then a linearly polarised harmonic gravitational wave with the waveforms:

$$\mathcal{A}_1(t) = h \sin \omega_g t \tag{83}$$

$$\mathcal{A}_2(t) = 0, \tag{84}$$

frequency $\omega_g/2\pi$ and dimensionless amplitude h. If the measurement time corresponds to n ($\in \mathbb{Z}^+$) cycles of this signal i.e. $\tau = \tau_n \overset{\text{def}}{=} 2n\pi/\omega_g \ll \tau_p$ then

$$\widetilde{\mathcal{A}}_1^{|\tau_n}(\omega) = \frac{h\,\omega_g}{\omega^2 - \omega_g^2}\left(e^{-i\omega\tau_n} - 1\right). \tag{85}$$

Hence $\widetilde{\mathcal{A}}_1^{|\tau_n}(\omega_p)$ has a maximum modulus at resonance when $\omega_g = \omega_p$, yielding

$$\widetilde{\mathcal{A}}_1^{|\tau_n}(\omega_g) = -\frac{ihn\pi}{\omega_g}. \tag{86}$$

In this case the signal-to-noise ratio (82) becomes

$$\frac{\mathcal{E}_p^{(S)}(\tau_n)}{\mathcal{E}_p^{(N)}(\tau_n)} \approx \frac{n\pi h^2 a_{1p}^2 Q_p}{4}\frac{EAL}{k_B T} \tag{87}$$

for an integer $n \ll Q_p/2\pi$. This formula is applicable to any 1-dimensional Cosserat structure with a specified reference configuration satisfying the material characteristics assumed above.

VI. A NARROW-BAND CIRCULAR LOOP GRAVITATIONAL ANTENNA

A circular reference configuration of a closed string may be parameterised with

$$x_0(s) = R\cos\frac{s}{R} \tag{88}$$

$$y_0(s) = R\sin\frac{s}{R} \tag{89}$$

$$z_0(s) = 0. \tag{90}$$

By using (7) this has a Frenet curvature

$$\kappa_0(s) \;=\; \frac{1}{R}. \tag{91}$$

The "shapes" of the associated normal modes satisfying (42) and (41) subject to periodic boundary conditions are given by

$$\psi_{[k,1]}(s) \;=\; \sqrt{2}\,\cos\frac{ks}{R} \tag{92}$$

$$\psi_{[k,2]}(s) \;=\; \sqrt{2}\,\sin\frac{ks}{R} \tag{93}$$

with multiple mode indices ($p = [k,1]$ or $[k,2]$ with positive integer k). The associated eigen-angular frequencies are

$$\omega_{[k,1]} = \omega_{[k,2]} \;=\; \frac{1}{R}\sqrt{(k^2+1)\,\frac{E}{\rho}}. \tag{94}$$

The effective modal masses, obtained by substituting the above into (54), are

$$m_{[k,1]} = m_{[k,2]} \;=\; 8\pi^3\rho A R\,(k^2+1). \tag{95}$$

It follows from (31) and (32) with the choice $\theta = 0$ that

$$\mathcal{F}_1(s) \;=\; -\frac{1}{2}\,\cos\frac{2s}{R} \tag{96}$$

$$\mathcal{F}_2(s) \;=\; -\frac{1}{2}\,\sin\frac{2s}{R}. \tag{97}$$

Substituting these into (46) and (47) we obtain the following overlap coefficients:

$$a_{1,[k,j]} \;=\; -\frac{1}{2\sqrt{2}}\,\delta_{2,k}\,\delta_{1,j} \tag{98}$$

$$a_{2,[k,j]} \;=\; -\frac{1}{2\sqrt{2}}\,\delta_{2,k}\,\delta_{2,j}. \tag{99}$$

Therefore only two quadrupole modes (corresponding to $p = [2,1]$ and $[2,2]$) are excited by the gravitational waves. The signal to thermal noise ratios for these two modes follow from (82) as

$$\frac{\mathcal{E}^{(S)}_{[2,j]}(\tau)}{\mathcal{E}^{(N)}_{[2,j]}(\tau)} \;\approx\; \frac{\pi\,\omega^2_{[2,j]}}{8}\,\frac{\tau_{[2,j]}}{\tau}\,\frac{EAR}{k_B T}\left|\widetilde{\mathcal{A}}^{|\tau}_j(\omega_{[2,j]})\right|^2 \tag{100}$$

where $j = 1, 2$. From (87)

$$\frac{\mathcal{E}^{(S)}_{[2,1]}(\tau_n)}{\mathcal{E}^{(N)}_{[2,1]}(\tau_n)} \approx \frac{n\pi^2 h^2 Q_{[2,1]}}{16} \frac{EAR}{k_B T}. \tag{101}$$

For $R = 1.8 \times 10^3$ m, $\rho = 8 \times 10^3$ kg/m^3, $E = 2.0 \times 10^{11}$ kg/m s^2, the quadrupole mode frequency is $\omega_{[2,1]}/2\pi = 1$ Hz. For a structure with a circular cross-section of radius $r = 0.01$ m the area $A = \pi r^2$. If one selects $T = 4$ K, $Q_{[2,1]} = 10^5$ and $n \approx 0.1 \times Q_{[2,1]}/2\pi$ then the condition $\mathcal{E}^{(S)}_{[2,1]}(\tau_n)/\mathcal{E}^{(N)}_{[2,1]}(\tau_n) = 1$ implies an amplitude sensitivity of $h \approx 2 \times 10^{-21}$ for such an antenna.

VII. A BROAD-BAND SPIRAL GRAVITATIONAL ANTENNA

By contrast to the narrow-band loop antenna in which only two quadrupole modes with the same frequency are excited by a harmonic gravitation wave as discussed in Section VI, we now demonstrate how a multi-frequency gravitational antenna may be constructed based on a spiral reference configuration. Consider an open string whose reference configuration lies on the x-y plane as before but with a non-constant Frenet curvature of the form

$$\kappa_0(s) = \frac{1}{R} \sqrt{\mu^3 \frac{s}{R} + 1} \tag{102}$$

for a constant μ and scale parameter R. It follows from (33) that

$$\mathcal{K}_0(s) = \frac{2}{3} \frac{R^3}{\mu^3} \left[\kappa_0(s)^3 - \kappa_0(0)^3 \right]. \tag{103}$$

Substituting (102) into (41) yields

$$\psi_p''(s) + \left(\frac{\mu^2}{R^2} \lambda_p - \frac{\mu^3}{R^3} s \right) \psi_p(s) = 0 \tag{104}$$

where

$$\lambda_p \stackrel{\text{def}}{=} \frac{1}{\mu^2} \left(\frac{\rho}{E} R^2 \omega_p^2 - 1 \right). \tag{105}$$

Solving (104) subject to tension-free boundary conditions yields

$$\psi_p(s) = A_p \operatorname{Ai}\left(\mu \frac{s}{R} - \lambda_p \right) + B_p \operatorname{Bi}\left(\mu \frac{s}{R} - \lambda_p \right) \tag{106}$$

and the condition

$$\begin{bmatrix} \mathrm{Ai}(-\lambda_p) & \mathrm{Bi}(-\lambda_p) \\ \mathrm{Ai}(\mu\ell - \lambda_p) & \mathrm{Bi}(\mu\ell - \lambda_p) \end{bmatrix} \begin{bmatrix} A_p \\ B_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{107}$$

where $\ell \overset{\text{def}}{=} L/R$ and the Ai, Bi are the standard Airy functions. The corresponding characteristic equation for λ_p,

$$\mathrm{Ai}(-\lambda_p)\,\mathrm{Bi}(\mu\ell - \lambda_p) - \mathrm{Ai}(\mu\ell - \lambda_p)\,\mathrm{Bi}(-\lambda_p) \;=\; 0, \tag{108}$$

determines the eigen-values λ_p. When comparing the spectrum of the eigen-frequencies of a spiral with that of a circular loop it is convenient to introduce the function

$$\Omega(s) \;\overset{\text{def}}{=}\; \kappa_0(s)\sqrt{5\,\frac{E}{\rho}}. \tag{109}$$

Note that for any $s \in [0, L]$ the spiral has a radius of curvature $1/\kappa_0(s)$ and from (94) it can be seen that $\Omega(s)$ equals the quadrupole mode angular frequency of a circular loop of radius $1/\kappa_0(s)$ with the same mass density ρ and Young's modulus E. Given values of λ_p as solutions of (108), it follows from (105) and (109) that the eigen-angular frequencies ω_p are given by

$$\omega_p \;=\; \Omega(0)\sqrt{\frac{\mu^2\,\lambda_p + 1}{5}} \tag{110}$$

and from (107)

$$B_p \;=\; -\frac{\mathrm{Ai}(\mu\ell - \lambda_p)}{\mathrm{Bi}(\mu\ell - \lambda_p)}\,A_p. \tag{111}$$

The normalisation condition (42) now takes the form

$$\int_0^\ell \left\{A_p\,\mathrm{Ai}\,(\mu\sigma - \lambda_p) + B_p\,\mathrm{Bi}\,(\mu\sigma - \lambda_p)\right\}^2 \mathrm{d}\sigma \;=\; \ell \tag{112}$$

where $\sigma \overset{\text{def}}{=} s/R$. Therefore (111) and (112) determine A_p and B_p. Furthermore the overlap coefficients a_{1p} and a_{2p} are obtained from (46) and (47) using (31), (32) and (106).

Finally from (87) the signal to (thermal) noise ratio for the p-th mode of the spiral may be written:

$$\frac{\mathcal{E}_p^{(S)}(\tau_n)}{\mathcal{E}_p^{(N)}(\tau_n)} \approx \frac{n\pi\ell h^2 a_{1p}^2 Q_p}{4} \frac{EAR}{k_B T}. \tag{113}$$

In Figure 4 we have plotted, for a given spiral geometry (determined by the parameters μ, ℓ and R, the values of $\hat{a}_p^2 = \ell(a_{1p}^2 + a_{2p}^2)$ for the first 60 normal modes of the spiral according to the equations above. The modes are distributed uniformly on the abscissa in terms of the non-dimensional eigen-angular frequencies $\hat{\omega}_p = \omega_p/\Omega(0)$. The broad-band response feature of this antenna is clearly visible in the region where significant signal overlap occurs.

For $\mu = 0.3$, $\ell = 16\pi$, $R = 1.9 \times 10^3$ m, $\rho = 8 \times 10^3$ kg/m^3, $E = 2.0 \times 10^{11}$ kg/m s^2, one finds for $p = 45$ that $\lambda_p = 95.5$ and $a_{1p} = 0.135$ corresponding to the eigen-frequency $\omega_p/2\pi = 1$ Hz. For a structure with a circular cross-section of radius $r = 0.01$ m the area $A = \pi r^2$. If one selects $T = 4$ K, $Q_p = 10^5$ and $n \approx 0.1 \times Q_p/2\pi$ then the condition $\mathcal{E}_p^{(S)}(\tau_n)/\mathcal{E}_p^{(N)}(\tau_n) = 1$ implies an amplitude sensitivity of $h \approx 2 \times 10^{-21}$ for such an antenna. Thus the spiral antenna maintains a sensitivity commensurate with that of the single loop antenna but with the added broad-band characteristic.

VIII. CONCLUSIONS

The simple Cosserat theory of rods has been outlined and used to estimate the thermal noise sensitivity of gravitational antennae constructed out of orbiting slender material structures. Although such calculations have not included many other competing noise sources we feel that they provide support for a novel concept. Orbiting Cosserat structures can accommodate both narrow-band and broad-band detectors and may offer a much cheaper alternative to existing space-based laser-interferometers. One may venture optimism that costs will become ever more competitive with the current rapid development of high-strength carbon-based fibres. Not withstanding economic considerations, the implementation of these ideas would be complimentary to existing global programmes of gravitational wave research. The design of detectors with optimised response in the 1 Hz spectral region would exploit a current niche in the existing gravitational antenna frequency spectrum and the detection of gravitational waves in this domain would provide vital information about stochastic backgrounds in the early Universe and the relevance of super-massive black holes to processes in the centre of galaxies.

Although we have concentrated on planar structures and short measurement times, in order to gain enhanced sensitivities, the extension to non-planar loops and spirals is in principle straightforward. For example, several "wire-balls" of approximately spherical shape may offer a viable method to monitor stochastic gravitational waves over longer periods of time.

There remain many further important issues that need detailed attention. However we believe that the use of *stochastic Cosserat thermo-mechanics* for slender rods in noisy gravitational fields offers a reliable research tool for the development of the scenarios outlined in this paper.

IX. ACKNOWLEDGEMENTS

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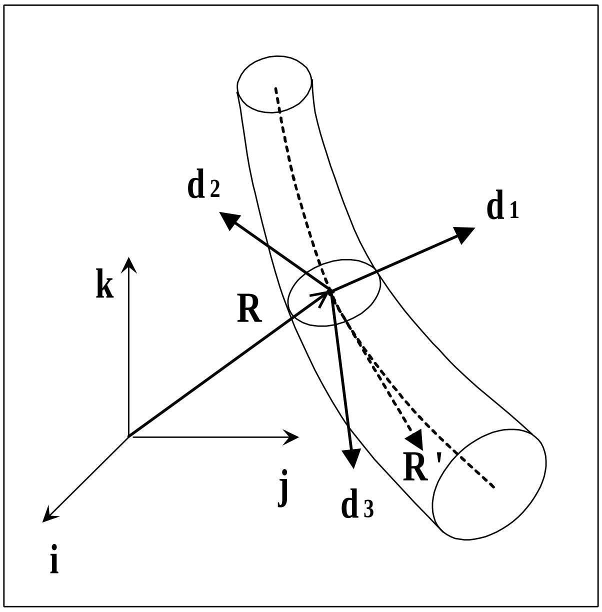

FIG. 1: Segment of a rod and the vectors that enter into its Cosserat description.

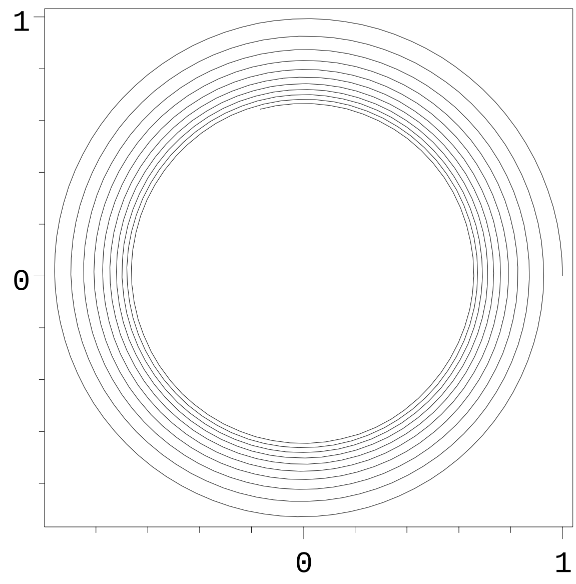

FIG. 2: Image of a spiral with $\mu = 0.3$ and $\ell = 16\,\pi$ plotted in the $\frac{x}{R}$ - $\frac{y}{R}$ plane. The radius of curvature of this spiral varies from $1/\kappa_0(0) = R$ at the outer end to $1/\kappa_0(L) = 0.65\,R$ at the inner end with approximately 8 ($\approx \ell/2\pi$) windings.

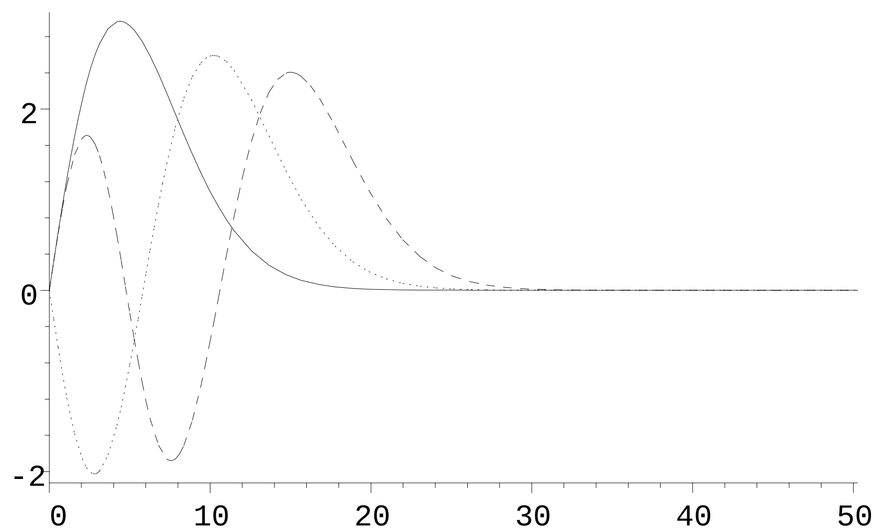

FIG. 3: Behaviour of the first 3 eigen-functions, $\psi_1(s)$ (solid), $\psi_2(s)$ (dotted) and $\psi_3(s)$ (dashed), for a spiral with $\mu = 0.3$ and $\ell = 16\,\pi$, plotted against $\sigma = s/R \in [0, \ell]$).

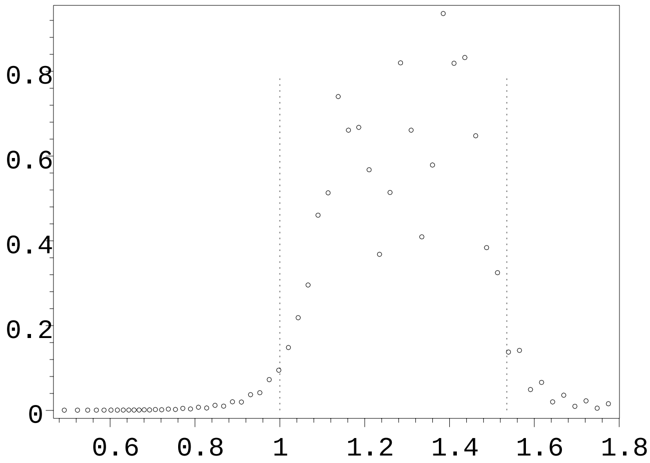

FIG. 4: The overlap parameters $\hat{a}_p^2 = \ell(a_{1p}^2 + a_{2p}^2)$ for the first 60 normal modes $(1 \leq p \leq 60)$ of a spiral with $\mu = 0.3$ and $\ell = 16\pi$ plotted against the non-dimensional angular frequency $\hat{\omega}_p = \omega_p/\Omega(0)$ calculated from λ_p using (110). Relatively larger $\hat{a}_p^2$ are obtained for $30 \lesssim p \lesssim 50$ corresponding to $1 \lesssim \hat{\omega}_p \lesssim \kappa_0(\ell)$=1.54 between the two vertical lines.